Phase-Sensitive Multimode Parametric Amplification in a Parabolic-Index Waveguide

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***Abstract*— We show that multiple spatial modes or images can be amplified by an optical parametric amplifier based on a graded-index waveguide, and that the modal gains can be equalized by using a superposition of several pump modes. We present a coupled-mode model of such an amplifier and develop gain equalization techniques for both 1- and 2-D cases. In the former case, we find that ∼20 modes can be amplified with a relatively low gain ripple by a superposition of four pump modes. In the latter case, over 250 modes can be similarly amplified by a superposition of $4 \times 4 = 16$ pump modes.**

***Index Terms*— Image amplification, mode-division multiplexing, multimode waveguides, parametric amplifiers.**

I. Introduction

THE fast-growing demand for optical link capacities [1] indicates that, within a decade, the capacity of single-mode fiber links with advanced modulation formats (e.g., multilevel quadrature amplitude modulation), even with the ideally distributed Raman amplification [2], [3], will approach Shannon limit, and future growth will rely on cost-effective increase either in the number of used fibers or fiber cores (space-division multiplexing, or SDM), or in the number of used fiber modes (mode-division multiplexing, or MDM). While current proof-of-concept MDM experiments (e.g., [4]) use separate single-mode optical amplifiers for each mode, the required scalability can only be achieved with an amplifier capable of handling the light from multiple cores or modes. The development of such an amplifier remains a big challenge. A 7-core erbium-doped fiber amplifier (EDFA) was recently demonstrated [5]. A-few-mode EDFAs was proposed [6], implemented [7], and used in transmission experiments (e.g., [8]), with the refractive-index, Er-doping, and pump profiles optimized to reduce the gain ripple among the modes. Such EDFAs, however, have drawbacks: multi-core amplifiers require precise alignment for efficient coupling (otherwise, their noise figure is very high [5]); the few-mode EDFAs suffer from random mode mixing owing to the environmental disturbances (e.g., bending and vibrations) and to the thermal variations in the spatial profiles of the refractive-index and gain, aggravated by the inhomogeneities in Er concentration. Thus, an amplifier is desired, which does not absorb the pump (e.g., parametric amplifier) and is environmentally robust (e.g., based on a rigid waveguide instead of flexible fiber). Mode-selective Raman gain was recently demonstrated and used to emphasize the fundamental mode of a multimode fiber link [9]. Self-imaging in a Raman-amplified step-index Si waveguide was proposed for mid-infrared image pre-amplification [10]. The $\chi^{(3)}$ nonlinearity, such as Raman effect, however, needs significant powers and adds further mode mixing due to intensity-dependent refractive index profile.

On the other hand, bulk-crystal-based optical parametric amplifiers (OPAs) in $\chi^{(2)}$ nonlinear media have been used as image amplifiers for long time [11]. The use of nonlinear waveguides and recycling of the pump via pump-reflecting waveguide facets can lead to required pump powers comparable to those of EDFAs. An additional advantage of an OPA is that it can operate as either phase-insensitive or phase-sensitive amplifier (PIA or PSA, respectively). In the former configuration, the OPA's noise figure (NF) is close to the quantum limit of 3 dB. In the latter, the NF can be 0 dB. Such noiseless amplification has been demonstrated with images [12] and time-domain signals [13]–[15], and has recently enabled ultra-low-noise fiber transmission links [16]. PSAs are particularly attractive for phase-encoded modulation formats, for which they can serve as phase regenerators [17], [18]. Therefore, it would be highly advantageous to use a spatially-broadband (i.e., multimode) OPA as a PSA in SDM or MDM transmission links. To enable such use, one needs to make the multimode OPA fiber-friendly and power-efficient by replacing the bulk-optics OPA with a waveguide-based device.

In this letter, we analyze such a spatially-multimode OPA in a $\chi^{(2)}$ nonlinear waveguide, with the goal of achieving equalized multimode gain with minimal mode mixing. The easiest waveguide to analyze is the one with infinitely-extending parabolic refractive index profile, for which analytical solutions in the form of Hermite-Gaussian (HG) and Laguerre-Gaussian (LG) modes are known [19]. Although this is a simplification of a real graded-index waveguide, the qualitative results and the intuitive approach to equalization that we develop are useful for practical waveguides as well. We build upon our recent work, in which we found [20] and expressed in compact basis [21] the independently-amplified eigenmodes of a bulk-optics OPA with an elliptic Gaussian pump, in the LIDAR image pre-amplification context [22], [23]. Such

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diagonalization is based on the simplectic nature of parametric interaction [24], which we previously exploited to find the eigenmodes of soliton's quantum noise [25], [26]. This letter is an extended version of our conference paper [27].

II. Model

Propagation of signal and idler waves in a $\chi^{(2)}$ medium obeys Helmholtz equation driven by a parametric term:

$$\nabla^2 E_{s,i}(\vec{\rho},z) + \frac{n_{s,i}^2(\vec{\rho})\omega_{s,i}^2}{c^2} E_{s,i}(\vec{\rho},z) = -4\frac{\omega_{s,i}^2 d_{\text{eff}}}{c^2} E_p(\vec{\rho},z) E_{i,s}^*(\vec{\rho},z). \quad (1)$$

We are looking for the solutions in the form $e_j(\vec{r},t) = E_j(\vec{\rho},z)e^{-i\omega_j t} + \text{c.c.}$, where $\vec{\rho}$ is a transverse vector with coordinates (x,y), z is the propagation direction, $n_j(\vec{\rho})$ is the refractive index profile, the intensity is given by $I_j(\vec{\rho},z) = 2\varepsilon_0 n_j c|E_j(\vec{\rho},z)|^2$ with index j taking values of s, i, or p, denoting the signal, idler, or pump field, respectively, and $\omega_p = \omega_s + \omega_i$. A frequency or polarization non-degenerate (phase-insensitive) OPA can be represented by a combination of two degenerate (phase-sensitive) OPAs [12], one for mode $A_+ = (E_s + E_i)/\sqrt{2} = X_+ + iY_+$, and the other for mode $A_- = (E_s - E_i)/\sqrt{2} = X_- + iY_-$, arranged in such a way that field quadratures X_+ and Y_- are amplified, while Y_+ and X_- are attenuated [28]. Thus, in this letter we consider only the phase-sensitive case, where signal and idler modes are degenerate ($s = i$), and $\omega_p = 2\omega_s$.

For a medium with $n^2(\vec{\rho}) = n_{\text{center}}^2[1 - 2\Delta(\rho/a)^2]$, an infinitely-extending parabolic index profile, where $\Delta = (n_{\text{center}}^2 - n_{\text{cladding}}^2)/2n_{\text{center}}^2 \approx (n_{\text{center}} - n_{\text{cladding}})/n_{\text{center}} << 1$, the eigenmodes of the linear version of Eq. (1) in a paraxial approximation are the well-known HG and LG modes [19]. In particular, the HG_{mn} eigenmodes have z-independent beam waists $a_{0j} = \sqrt{\lambda_j b/[4\pi^2 n_{\text{center}}(\lambda_j)]}$ and propagation constants

$$\beta_{mn}^j = \sqrt{k_j^2 - k_j\frac{4\pi}{b}(m+n+1)} \approx k_j - \frac{2\pi}{b}(m+n+1), \quad (2)$$

where $k_j = n_{\text{center}}(\lambda_j)\omega_j/c$, $b = \pi a\sqrt{2/\Delta}$ is the beat length, and index j takes values of either s or p. Despite different propagation constants, the modes of all orders periodically come to a point where their relative phases are the same as those at the waveguide input (up to a multiple of 2π), and the spatial period between these points is b (beat length). This means that the parabolic-index waveguide produces an image of its input periodically with period b.

Now we consider the waveguide to be nonlinear and pumped by an HG pump beam of the order m_p, n_p with power $P_{m_p n_p}$. We project the right-hand side of Eq. (1) onto the signal HG basis, which involves an overlap integral between pump mode of order m_p and the signal modes of orders m, m' coupled by the pump in one dimension, as well as another overlap integral between pump mode of order n_p and the signal modes of orders n, n' coupled by the pump in the other dimension. These overlap integrals are non-zero only for sets of m_p, m, m' and n_p, n, n' satisfying the "triangle rule": none of the three numbers m_p, m, m' and n_p, n, n' is greater than the sum of the other two, i.e., they can form sides of a triangle [29]. For each set, the wavevector mismatch $\Delta\beta = \beta_{m_p n_p}^p - \beta_{mn}^s - \beta_{m'n'}^s$ is

$$\Delta\beta \approx k_p - 2k_s + \frac{2\pi}{b} - \frac{2\pi}{b}(m_p + n_p - m - n - m' - n'). \quad (3)$$

Under condition that $k_p - 2k_s = -2\pi/b$ (which, e.g., can be satisfied by type-I phase matching), the "triangle rule" combined with requirement of perfect phase matching leads to coupling only between the signal mode orders m and m' satisfying $m_p = m + m'$, as well as n and n' satisfying $n_p = n + n'$. Here we assume the parametric gain over one beat length b to be small (i.e., even the least non-zero $\Delta\beta$ greatly dominates over gain), hence only the perfectly phase-matched interaction results in noticeable gain. Then, the partial differential Eq. (1) is reduced to a system of ordinary differential equations (coupled-mode equations) for signal mode amplitudes A_{mn}:

$$\mathrm{d}A_{mn}(z)/\mathrm{d}z = \sum_{m',n'}\sum_{m_p,n_p} \kappa_{m_p n_p} B_{mm'}^{m_p} B_{nn'}^{n_p} A_{m'n'}^*(z), \quad (4)$$

where $\kappa_{m_p n_p} = i\exp(i\theta_{m_p n_p})[\omega_s^2 d_{\text{eff}}^2 P_{m_p n_p}/(2\varepsilon_0 n_s^3 c^3 \pi a_{0p}^2)]^{1/2}$,

$$B_{mm'}^{m_p} = \delta_{m_p, m+m'}\{m_p!/[2^{m_p} m!\,(m_p - m)!]\}^{1/2}, \quad (5)$$

and we have added the possibility of pump beam containing superposition of various mode orders m_p,n_p. The mode-coupling coefficient $B_{mm'}^{m_p}$ (based on the overlap integral) at large m_p can be approximated by a Gaussian $B_{mm'}^{m_p} \approx \delta_{m_p,m+m'}[2/(\pi m_p)]^{1/4}\exp\left[-(m - m_p/2)^2/m_p\right]$ centered at $m = m_p/2$ with FWHM $= 2(m_p \ln 2)^{1/2} \approx 1.67 m_p^{1/2}$.

If the waveguide is multimode in only one dimension, then Eq. (4) can be reduced to

$$\mathrm{d}A_m(z)/\mathrm{d}z = \sum_{m'}\sum_{m_p} \kappa_{m_p} B_{mm'}^{m_p} A_{m'}^*(z), \quad (6)$$

or

$$\mathrm{d}A_m(z)/\mathrm{d}z = \sum_{m'} \gamma_{mm'} A_{m'}^*(z), \quad (7)$$

where $\gamma_{mm'} = \sum_{m_p} \kappa_{m_p} B_{mm'}^{m_p}$. Assuming no pump depletion and extending our PSA modeling approach [20], [21], [24], we find the independently amplified PSA modes (eigenmodes) by diagonalizing the coupling tensor of Eq. (4) or coupling matrix of Eq. (7). The spectrum of the matrix represents the spectrum of "gain per unit length" values for various eigenmodes.

The pump saturation in an OPA with more than one signal leads to a strong crosstalk and should be avoided. This justifies our undepleted-pump approximation. The saturation-caused crosstalk can be reduced in an OPA with a pump cavity, but the mode composition of the depleted pump would vary over distance, and such a problem is outside of this work's scope.

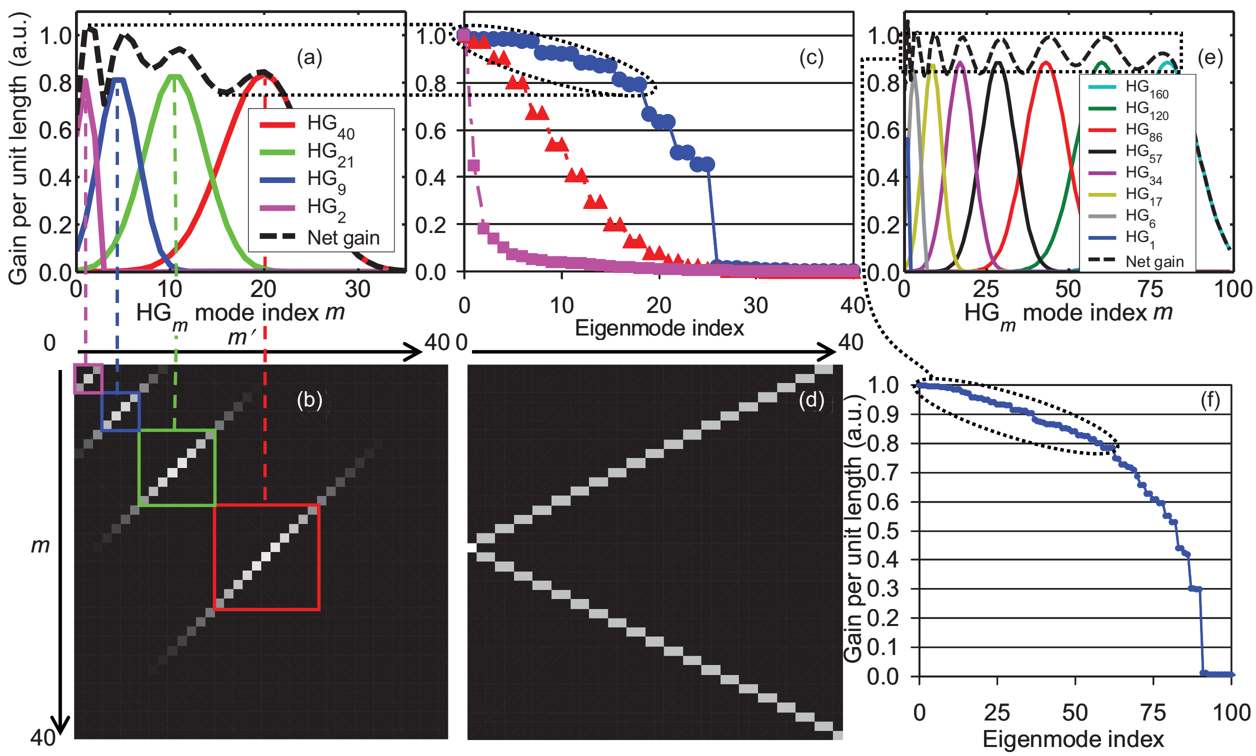

Fig. 1. 1-D parametric amplifier. (a) Individual pump contributions γ_{m,m_p-m} from four HG_{m_p} pumps with $m_p = 2$, 9, 21, 40, and the net gain obtained by adding the individual contributions. (b) Grayscale map of the coupling matrix $\gamma_{mm'}$ for the four-pump case of (a). (c) Eigenvalue spectra (the mode index starts at 0) for the amplifiers pumped by superpositions of either one ($m_p = 40$; dashed red line with triangles), four ($m_p = 2$, 9, 21, 40; solid blue line with circles), or 41 ($m_p = 0...40$; dot-dashed pink line with squares) HG pump orders. (d) Grayscale map of the PSA eigenmodes in HG representation for the $m_p = 40$ case. (e) Individual pump contributions γ_{m,m_p-m} from eight HG_{m_p} pumps with $m_p = 1$, 6, 17, 34, 57, 86, 120, 160, and the net gain obtained by adding the individual contributions. (f) Eigenvalue spectrum (the mode index starts at 0) for the amplifier pumped by the eight-pump superposition of (e). Vertical scales of (a), (c), and (e) are the same; vertical scales of (b) and (d) are the same. Horizontal scales of (c) and (d) are the same. Dotted lines relate the regions of low gain ripple in (a) and (e) to the flat portions of the eigenvalue spectra in (c) and (f), respectively. Each eigenvalue spectrum in (c) and (f) is normalized by its largest eigenvalue. The squares in (b) indicate mode ranges that have noticeable gains contributed by only one pump – for the HG modes in each of these squares the four-pump PSA behaves close to a single-pump PSA, coupling a mode m to a single mode $m_p - m$.

III. Results and Discussion

Let us consider the one-dimensional (1-D) case first. When the PSA is pumped by one pump mode m_p at a time, the PSA eigenmodes are the symmetric and anti-symmetric superpositions of signal modes HG_m and HG_{m_p-m}, with eigenvalues closely following the overlap coefficients $B^{m_p}_{mm'}$. For the case of $m_p = 40$, the eigenmodes are shown in Fig. 1(d) and the corresponding eigenvalues – in Fig. 1(c) (dashed red line with triangles). A quick examination of the eigenvalue spectrum for a combination of a few pump modes, regardless of their phases, always shows that for closely spaced pump orders the spectrum decays very fast, i.e., the most amplified eigenmode has a very large gain, whereas even the next eigenmode already has a significantly lower gain [this is illustrated by dot-dashed pink line with squares in Fig. 1(c) for an extreme case of 41 closely-spaced pump modes]. This situation is undesirable, because it results in large gain ripple across the signal modes. After a thorough optimization, we have found that the flattest eigenmode spectrum (and correspondingly smallest gain ripple) is obtained by spacing the pump orders so that the gain curves for the signal modes HG_m (under $m' = m_p - m$ condition) contributed by various pump orders intersect at ∼50% of their maximum value [see Fig. 1(a)]. When one looks at the coupling matrix $\gamma_{mm'}$ [see Fig. 1(b)], this condition corresponds to the situation where modal regions of noticeable gains from various pumps [colored squares in Fig. 1(b)] do not overlap, i.e., modes within each region behave as in a single-pump PSA. Figure 1(a) shows the partial gain for various signal modes HG_m produced by the individual pump modes with orders $m_p = 2$, 9, 21, and 40, as well as the net gain produced by their superposition. It is easy to see that a relatively flat gain for ∼20 signal modes is achievable by combining these 4 pump modes. This is confirmed by the corresponding eigenvalue spectrum [solid blue line with circles in Fig. 1(c)], with gain for ∼20 eigenmodes within 20% of maximum. With 8 pump modes ($m_p = 1$, 6, 17, 34, 57, 86, 120, 160), the flat gain region extends to ∼60 modes, as can be seen from Figs. 1(e) and (f). The input phases of different pump orders can be chosen arbitrarily; however, it is convenient to make them equal, because in that case all the signal modes have the same optimum input phase.

The two-dimensional (2-D) case behaves similarly to 1-D, even though its coupling tensor is more difficult to visualize. Figure 2 shows the eigenvalue spectra in the 2-D case with either 4 or 16 pump modes, whose orders m_p and n_p are spaced similarly to the 1-D case. We see that with just 4 pump modes one can amplify >30 signal modes, and hundreds of

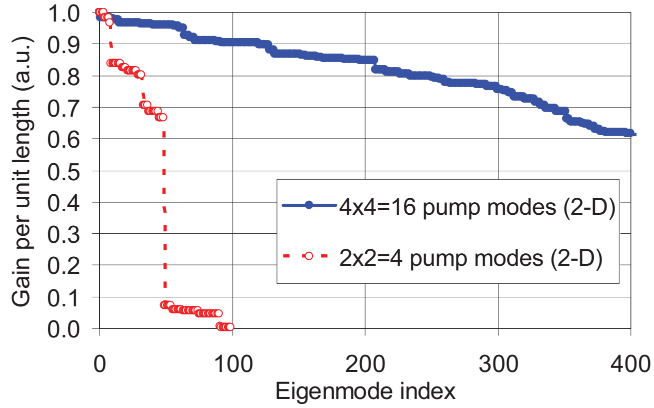

Fig. 2. Eigenvalue spectra (the mode index starts at 0) for 2-D parametric amplifiers pumped by superpositions of either two ($m_p = 2, 9$; dashed red line with empty circles) or four ($m_p = 2, 9, 21, 40$; solid blue line with filled circles) HG pump orders in each dimension. Both spectra are normalized by their largest eigenvalues.

signal modes are amplified with similar gains by a 16-mode pump. Of course, a real graded-index waveguide with finite cladding diameter may not support so many modes. In that case, our result means that, with just a few pump modes, flat gain can be achieved over the entire set of waveguide-supported modes.

One should note that, if expansion over LG_{pl} rather than HG_{mn} modes is used, the mode selection rules remain simple: a) $l_p = l + l'$ (orbital momentum conservation) and b) $2(p_p - p - p') + |l_p| - |l| - |l'| = 0$ (phase matching). However, the overlap integral formulas are more complex and not conducive to intuitive interpretation, which makes HG analysis preferable.

IV. Conclusion

We have shown that multimode optical parametric amplifier for SDM or MDM communication can be realized in a graded-index waveguide with parabolic refractive-index profile, written in a $\chi^{(2)}$ nonlinear material, e.g., $LiNbO_3$ (with properly customized titanium diffusion process) pumped by second harmonic of a fiber laser. We have developed a mode-coupling model for such an amplifier and shown that in 1-D case it can support ~20 modes with 4-mode pumping and ~60 modes with 8-mode pumping, and in 2-D case it can support over 30 modes with 2×2-mode pumping and over 250 modes with 4×4-mode pumping, each with relative gain ripple (dB of ripple per dB of gain) of ±10%. This makes the waveguide-based parametric amplifier promising both for SDM or MDM optical communication and for faint image pre-amplification, with pump power savings over the bulk-crystal amplifier.

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